The Mathematics Of Stock Option Valuation - Part Three
Solution Via Risk-Neutral Probabilities

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In Part One we explained why valuing a call option as a stand-alone asset using risk-adjusted discount rates will almost always lead to an incorrect value because the value determined in this manner will most likely be subject to arbitrage. In Part Two we calculated the no-arbitrage price of a call option via partial differential equations. In this section we will calculate the no-arbitrage price of a call option via risk-neutral probabilities.

Risk-averse investors demand a rate of return on risky assets that is greater than the risk-free rate. Risk in this context is defined as uncertainty. Under the actual probability measure (The P Measure) the value of a risky asset today will be less than the asset's expected value tomorrow discounted at the risk-free rate. If S_0 is the price of a risky asset today, S_1 is the uncertain payoff on this asset in one year and r is the risk-free rate then...

$$S_0 < (1+r)^{-1}\,\mathbb{E}^P\Bigg[S_1\Bigg] \tag{1}$$

The above relationship holds because the discount rate applicable to this asset under the actual probability measure is comprised of the risk-free rate plus a risk premium. We will show that in a complete market all assets prices are martingales such that under the risk-neutral probability measure (The Q Measure) the value of a risky asset today will be equal to the asset's expected value tomorrow discounted at the risk-free rate. Under the risk-neutral probability measure equation (1) above becomes...

$$S_0 = (1+r)^{-1}\,\mathbb{E}^Q\Bigg[S_1\Bigg] \tag{2}$$

The probabilities under Measure Q incorporate the risk premium that is included in the discount rate under Measure P. In effect what is happening is that the risk-neutral probability for a good outcome is less that the actual probability and the risk-neutral probability for a bad outcome is greater than the actual probability. The actual probabilities are adjusted such that the investor is in effect *risk neutral.*

Legend of Symbols

B_t	=	Zero-coupon bond price at time t
C_t	=	Call option price at time t
M_t	=	Price of an Arrow-Dubreu security that pays \$1 in state w_u and \$0 in w_d
N_t	=	Price of an Arrow-Dubreu security that pays \$1 in state w_d and \$0 in w_u
S_t	=	Stock price at time t
r	=	Risk-free rate of interest
t	=	Time period in years

The One Period Economy (From Part One)

Our economy has two states of the world at time $t=1$. In state w_u the stock price moves up to \$180 and in state w_d the stock price moves down to \$80. We have a call option on this stock with an exercise price of \$120 that can be exercised at period $t=1$. If the stock price is above the exercise price the option will be exercised, otherwise it will be allowed to expire unexercised. We also have a risk-free bond that pays one dollar in both states of the world, an Arrow-Dubreu security that pays one dollar in state w_u and nothing in state w_d and an Arrow-Dubreu security that pays one dollar in state w_d and nothing in state w_u. The table below presents our one period economy and the two states of the world at time $t = 1$...

The One Period Economy:

State of the world	w_u	w_d
Stock price (S_1)	\$180	\$80
Call price (C_1)	60	0
Risk-free bond price (B_1)	1	1
Arrow-Dubreu price (M_1)	1	0
Arrow-Dubreu price (N_1)	0	1

We currently sit at $t = 0$ where the state-of-the-world at $t = 1$ is unknown. The risk-free rate during time period one is 5.00% and the actual (real-world) probabilities of states w_u and w_d are 0.50 and 0.50, respectively.

Arrow-Debreu Securities

A complete market is one in which for every state of the world there is a combination of traded assets that is equivalent to a pure contingent claim, which is a security that pays one dollar in a particular state of the world and nothing otherwise. Such a security is a called an Arrow-Debreu security. The complete market assumption allows us to replicate any possible payoff with a linear combination of these securities. The existance of Arrow-Debreu securities, which is guaranteed under the complete markets assumption, allows us to derive and utilize risk-neutral probabilities.

A bond that pays one dollar in state w_u and one dollar in state w_d has an expected payoff at time $t = 1$ of one dollar. Since this bond is risk-free (the payoff is one dollar regardless of the state of the world) the price of this bond at time $t = 0$ is the payoff at time $t = 1$ discounted at the risk-free rate. The equation for the price of the risk-free bond is...

$$\begin{aligned} B_0 &= 1.00 \times (1+r)^{-1} \\ &= 1.00 \times 1.05^{-1} \\ &= 0.95238 \end{aligned} \tag{3}$$

A portfolio of one Arrow-Debreu security that pays one dollar in state w_u (and nothing in state w_d) and one Arrow-Debreu security that pays one dollar in state w_d (and nothing in state w_u) perfectly replicates the payoff of the risk-free bond in equation (3) above. It follows that the time $t = 0$ price of a portfolio of one M_0 and one N_0 should equal the time $t = 0$ price of the risk-free bond B_0. If the prices were not equal then an arbitrage exists. The mathematical equivalent of this relationship is...

$$\begin{aligned} M_0 + N_0 &= B_0 \\ &= 0.95238 \end{aligned} \tag{4}$$

If the price of a portfolio of one M_0 and one N_0 equals the price of a risk-free bond at time $t = 0$ then this portfolio times the risk-free rate should equal the price of the risk-free bond at time $t = 1$. If the prices were not equal then an arbitrage exists. The mathematical equivalent of this relationship is...

$$\begin{aligned} M_0 + N_0 &= B_0 \\ (1+r) \times (M_0 + N_0) &= (1+r) \times B_0 \\ (1+r)M_0 + (1+r)N_0) &= 1.00 \end{aligned} \tag{5}$$

We will make the following definitions related to equation (5) above...

$$\hat{p} = (1+r)M_0 \tag{6}$$

$$\hat{q} = (1+r)N_0 \tag{7}$$

Such that equation (5) becomes...

$$\hat{p} + \hat{q} = 1.00 \tag{8}$$

Risk-Neutral Probabilities

In Part I we determined that the price of our stock at time $t=0$ was \$100. The stock was valued under the actual probability measure (Measure P where the probability of state w_u is 0.50 and the probability of state w_d is 0.50). The stock price at time $t=0$ under this measure is the expected payoff at time $t=1$ discounted at the risk-adjusted discount rate, which was 30%. The equation for stock price under the actual probability measure where p is the probability of state w_u and q is the probability of state w_d is...

$$\begin{aligned} S_0 &= (1+k)^{-1}\,\mathbb{E}^P\Bigg[S_1\Bigg] \\ &= (1+k)^{-1}\Bigg[180\,p+80\,q\Bigg] \\ &= 1.30^{-1}\Bigg[(180)(0.50)+(80)(0.50)\Bigg] \\ &= 100.00 \end{aligned} \tag{9}$$

The payoffs on the stock at time $t=1$ can be perfectly replicated via a portfolio of Arrow-Debreu securities. The replicating portfolio would be structured such that the payoff on the portfolio at time $t=1$ would equal \$180 in state w_u and \$80 in state w_d. The price of the stock at time $t=0$ is therefore the cost incurred to build the replicating portfolio at time $t=0$. The equation for stock price at time $t=0$ as a function of a portfolio of Arrow-Debreu securities is...

$$S_0 = 180M_0 + 80N_0 \tag{10}$$

We will now multiply equation (10) above by one plus the risk-free rate. After performing this operation equation (10) becomes...

$$\begin{aligned} S_0 &= 180M_0 + 80N_0 \\ (1+r)S_0 &= 180(1+r)M_0 + 80(1+r)N_0 \end{aligned} \tag{11}$$

We will now note that equation (6) and equation (7) can be substituted into equation (11) above. After making these substitutions equation (11) becomes...

$$\begin{aligned} (1+r)S_0 &= 180(1+r)M_0 + 80(1+r)N_0 \\ (1+r)S_0 &= 180\,p + 80\,q \\ S_0 &= (1+r)^{-1}(180\,\hat{p} + 80\,\hat{q}) \end{aligned} \tag{12}$$

After noting that $\hat{q} = 1-\hat{p}$ we can solve for the variable $\hat{p}$ in equation (12) above. The value of $\hat{p}$ is...

$$\begin{aligned} S_0 &= (1+r)^{-1}[180\,\hat{p} + 80\,\hat{q}] \\ 100.00 &= 1.05^{-1}[180\,\hat{p} + 80\,\hat{q}] \\ 100.00 &= (171.43)(\hat{p}) + (76.19)(1-\hat{p}) \\ 23.81 &= 95.24\hat{p} \\ \hat{p} &= 0.25 \end{aligned} \tag{13}$$

We will now solve for the variable $\hat{q}$ in equation (12) above. The value of $\hat{q}$ is...

$$\begin{aligned} \hat{p} + \hat{q} &= 1.00 \\ \hat{q} &= 1.00 - \hat{p} \\ \hat{q} &= 1.00 - 0.25 \\ \hat{q} &= 0.75 \end{aligned} \tag{14}$$

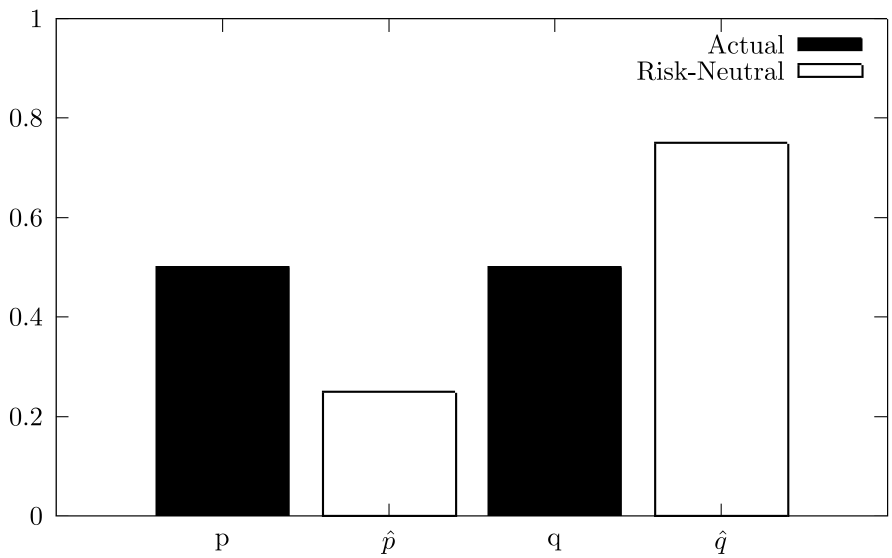
We will call $\hat{p}$ in equation (13) and $\hat{q}$ in equation (14) the Risk-Neutral probability distribution and p and q the Actual probability distribution. We can view $\hat{p}$ and $\hat{q}$ as a probability distribution because...

1 $\hat{p} + \hat{q} = 1$
2 $0 \le \hat{p} \le 1$
3 $0 \le \hat{q} \le 1$

The Risk-Neutral probability distribution $\hat{p}$ and $\hat{q}$ is equivalent to the Actual probability distribution p and q because...

1 Both probability distributions agree on what values are possible (probabilities are greater than zero)
2 Both probability distributions agree on what values are impossible (probabilities are equal to zero)

Both the Risk-Neutral and Actual probability distributions are graphed below. Note that the Risk-Neutral probability distribution puts less weight on the good outcome (the risk-neutral probability $\hat{p}$ is less than the actual probability p) and more weight on the bad outcome (the risk-neutral probability $\hat{q}$ is greater than the actual probability q). The reason for this disparity is that the risk-premium is reflected in the risk-neutral probabilities and not in the actual probabilities.



We can now state that in a complete market all asset prices are martingales in that the asset's expected future value under the risk-neutral measure discounted at the risk-free rate equals the asset's price today. We can say this because if this were not so then an arbitrage exists. We can now restate the stock price equation (9) above as...

$$
\begin{aligned}
S_0 &= (1+r)^{-1}\,\mathbb{E}^Q\Bigg[S_1\Bigg] \\
&= 1.05^{-1}\Bigg[(180)(0.25)+(80)(0.75)\Bigg[\\
&= 100.00
\end{aligned}
\tag{15}
$$

Call Price Under the Risk-Neutral Probability Measure

Since all asset prices are martingales under the risk-neutral measure the price of our call option must also be a martingale under this measure. The price of the call option today under the risk-neutral measure is...

$$
\begin{aligned}
C_0 &= (1+r)^{-1}\,\mathbb{E}^Q\Bigg[C_1\Bigg] \\
&= 1.05^{-1}\Bigg[(60)(0.25)+(0)(0.75)\Bigg[\\
&= 14.29
\end{aligned}
\tag{16}
$$

The call option value was determined to be \$14.34 in Part II where we valued the call via partial differential equations. In this section we valued the call using risk-neutral probabilities and got a value of \$14.29. The difference in values is due to rounding.